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FINITE GROUPS THAT ARE PRODUCTS OF TWO NORMAL SUPERSOLUBLE SUBGROUPS

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Abstract: Let $\mathfrak{P}\imath$ be the class of all finite groups that are products of two normal supersoluble subgroups. Let $\mathfrak{P}$ be the class of all nonsupersoluble $\mathfrak{P}\imath$-groups G such that all proper $\mathfrak{P}\imath$-subgroups of G and nontrivial factor groups of G are supersoluble. We classify the $\mathfrak{P}$-groups.

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1. Introduction

Throughout the paper, all groups are finite, G always denotes a finite group, p denotes a prime, and F_p denotes the finite field of elements p. Let $|G|$ denote the order of G and let $\pi(G)$ denote the set of all primes dividing $|G|$. A group G is said to be *p-closed* if G has a normal Sylow p-subgroup.

It is well known that the product of two normal soluble subgroups is soluble and the product of two normal nilpotent subgroups is nilpotent; but the product of two normal supersoluble subgroups is not necessarily supersoluble. Therefore, the following problems arise:

Problem 1.1. *Under which conditions will the product of two* (*normal*) *supersoluble subgroups be supersoluble still?*

Problem 1.2 (see [1, Chapter II, Problem 6.34]). *What is the structure of the nonsupersoluble group that is a product of two* (*normal*) *supersoluble subgroups?*

Many authors have studied Problem 1.1, for example, in [2–14]. In particular, Baer proved that if G is the product of two normal supersoluble subgroups and $[G, G]$ is nilpotent, then G is supersoluble [2]; Friesen proved that G the product of two normal supersoluble subgroups M and N is supersoluble if the indexes $|G : M|$ and $|G : N|$ are coprime [3].

Recently, Guo and Kondrat′ev in [4] and [5] described the minimal nonsupersoluble group that is the product of two normal supersoluble subgroups. They proved that a finite minimal nonsupersoluble group G is decomposable into the product of two normal supersoluble subgroups if and only if $G/F(G)$ is a primary minimal nonabelian group.

Let $\mathfrak{P}\imath$ be the class of all groups that are the product of two normal supersoluble subgroups. Let $\mathfrak{P}$ be the class of all nonsupersoluble $\mathfrak{P}\imath$-groups G such that all proper $\mathfrak{P}\imath$-subgroups and nontrivial factor groups of G are supersoluble. Naturally, the following problem arises:

Problem 1.3. *What is the structure of a* $\mathfrak{P}$*-group?*

The main purpose of this paper is to resolve Problem 1.3. In fact, we give the classification of $\mathfrak{P}$-groups (see Theorem 3.6 below). As a consequence, we also obtain a necessary and sufficient condition for a $\mathfrak{P}\imath$-group to be supersoluble (see Corollary 4.1).

All unexplained notation and terminology are standard. The reader is referred to [15, 16] if need be.

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2. Preliminaries

We will use the following available results.

Lemma 2.1 (see [17, Theorem 1.8.17]). *Suppose that M is a normal nilpotent subgroup of G and $M \cap \Phi(G) = 1$. Then M is the direct product of some minimal normal subgroups of G.*

Recall that G is a Frobenius group with complement H if H is a nontrivial proper subgroup of G such that $H \cap H^g = 1$ for all $g \in G\backslash H$ (see [18, Chapter V, Definition 8.1]). In this case,

$$K = \Big(G\backslash\Big(\bigcup_{g\in G} H^g\Big)\Big) \cup \{1\}$$

is a normal subgroup of G, and K is called the *Frobenius kernel* of G corresponding to the complement H. Moreover, $G = K \rtimes H$.

Lemma 2.2 (see [18, Chapter V, Theorem 8.7]). *Suppose that $G = KH$ is a Frobenius group with kernel K and complement H. If $p > 2$, then the Sylow p-subgroups of H are cyclic; if $p = 2$, then the Sylow p-subgroups of H are either cyclic groups or a generalized quaternion group. In particular, if H is abelian then H is cyclic.*

Recall that a group A acts regularly on G if A is nontrivial and $C_G(\langle a\rangle) = \{g \mid g \in G,\ g^a = g\} = 1$ for every $1 \neq a \in A$. If A is a group of automorphisms of G and acts regularly on G, then A is called a *regular group of automorphisms* of G.

The following lemma is an exercise of Gorenstein's book [16, Chapter 2, Exercise 17]. For the convenience of the reader, we include a proof.

Lemma 2.3. *If A is a regular group of automorphisms of G, then the semidirect product $G \rtimes A$ is a Frobenius group with kernel G and complement A.*

Proof. For each $ga \in G \rtimes A\backslash A$, where $g \in G$ and $a \in A$, it is clear that $g \neq 1$. Assume that $A \cap A^{ga} \neq 1$. Then there exists $1 \neq b \in A \cap A^{ga}$. It follows that there is $c \in A$ such that $b = c^{ga}$; i.e., $b = a^{-1}g^{-1}cga$. This implies that $g^c = g$. Therefore, $c = 1$ since $g \neq 1$ and A acts regularly on G. Hence $b = 1$; a contradiction. $A \cap A^{ga} = 1$ for all $ga \in G \rtimes A\backslash A$. This implies that $G \rtimes A$ is a Frobenius group with complement A. Since G is a normal complement of A in $G \rtimes A$, G is the Frobenius kernel of $G \rtimes A$. $\square$

3. The Main Result

Let $E_p(1,1,1) = \langle a, b \mid c = [a,b],\ a^p = b^p = c^p = [a,c] = [b,c] = 1\rangle$, $p > 2$, and $M_p(n,1) = \langle a, b \mid a^{p^n} = b^p = 1,\ a^b = a^{1+p^{n-1}}\rangle$, $n \geq 2$. Note that every maximal subgroup of $M_p(n,1)$ contains $\langle a^p\rangle = Z(M_p(n,1))$ as a maximal subgroup. Then $M_p(n,1)$ is a minimal nonabelian group.

Firstly we construct four kinds of groups that are $\mathfrak{P}$-groups, and it will be shown that these groups are the only $\mathfrak{P}$-groups in Theorem 3.6.

Example 3.1. Let p and q be two primes with $q \mid p-1$ and $q > 2$ and assume that $P = \mathrm{F}_p v_1 + \mathrm{F}_p v_2 + \cdots + \mathrm{F}_p v_q$ is a q-dimensional vector space over F_p. Let ω be a primitive qth root of unity in F_p. Let

$$\alpha = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega \end{pmatrix}_{q\times q}, \quad \beta = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{q-1} \end{pmatrix}_{q\times q},$$

$$\gamma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{q\times q}$$

and $Q = \langle \alpha, \beta, \gamma \rangle$ where Q is generated by α, β, and γ with matrix multiplication. Then $Q \simeq E_q(1,1,1)$. Put $G = P \rtimes Q$, $M = P \rtimes \langle \alpha, \beta \rangle$, and $N = P \rtimes \langle \alpha, \gamma \rangle$. Then M and N are normal supersoluble subgroups of G and $G = MN$. But G is nonsupersoluble. We denote G by $E(p,q,1)$.

EXAMPLE 3.2. Let p and q be two primes with $q^n \mid p-1$ and $n \geq 2$. Suppose that $P = \mathrm{F}_p v_1 + \mathrm{F}_p v_2 + \cdots + \mathrm{F}_p v_q$ is a q-dimensional vector space over F_p. Let ω be a primitive qth root of unity and let θ be a primitive q^nth root of unity in F_p. Let

$$\beta = \begin{pmatrix} \theta & & & \\ & \theta\omega^1 & & \\ & & \ddots & \\ & & & \theta\omega^{q-1} \end{pmatrix}_{q\times q}, \quad \gamma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{q\times q}$$

and $Q = \langle \beta, \gamma \rangle$. Then $Q \simeq M_q(n,1)$. So, $G = P \rtimes Q$, $M = P \rtimes \langle \beta \rangle$, and $N = P \rtimes \langle \beta^q, \gamma \rangle$. Now, M and N are normal supersoluble subgroups of G and $G = MN$. Clearly G is nonsupersoluble. We denote G by $M(p,q,n,\theta,\omega)$.

EXAMPLE 3.3. Let p be a prime with $4 \mid p-1$ and let $P = \mathrm{F}_p v_1 + \mathrm{F}_p v_2$ be a 2-dimensional vector space over F_p. Let θ be a primitive 4th root of unity in F_p. Let

$$\beta = \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $Q = \langle \beta, \gamma \rangle$. Then $Q \simeq Q_8$. Let $G = P \rtimes Q$, $M = P \rtimes \langle \beta \rangle$, and $N = P \rtimes \langle \gamma \rangle$. Then M and N are normal supersoluble subgroups of G and $G = MN$. But G is a nonsupersoluble Frobenius group with kernel P and complement Q. We denote G by $Q(p,2)$.

EXAMPLE 3.4. Let p be a prime with $4 \nmid p-1$ and let $P = \mathrm{F}_p v_1 + \mathrm{F}_p v_2$ be a 2-dimensional vector space over F_p. Let

$$\alpha = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $Q = \langle \alpha, \beta, \gamma \rangle$. Then $Q \simeq D_4$. Let $G = P \rtimes Q$, $M = P \rtimes \langle \alpha, \beta \rangle$, and $N = P \rtimes \langle \alpha, \gamma \rangle$. Then M and N are normal supersoluble subgroups of G and $G = MN$. Clearly, G is nonsupersoluble. We denote G by $D(p,2)$.

$E(p,q,1)$, $M(p,q,n,\theta,\omega)$, $Q(p,2)$, and $D(p,2)$ are $\mathfrak{P}$-groups. In fact, on the one hand P is the unique minimal normal subgroup of $E(p,q,1)$, $M(p,q,n,\theta,\omega)$, $Q(p,2)$, and $D(p,2)$. This implies that every nontrivial factor group of $E(p,q,1)$, $M(p,q,n,\theta,\omega)$, $Q(p,2)$, and $D(p,2)$ is supersoluble. On the other hand, $E(p,q,1)$, $M(p,q,n,\theta,\omega)$, and $Q(p,2)$ are minimal nonsupersoluble since every proper subgroup of Q is an abelian group with exponent dividing $p-1$. (Note that if $G = PQ$ where P is a normal abelian Sylow p-subgroup of G with $C_G(P) = P$ and Q is a Sylow q-subgroup of G with $q \neq p$, then G is supersoluble if and only if Q is an abelian group with exponent dividing $p-1$.) For $D(p,2)$, since Q has the unique proper subgroup $A = \langle \beta\gamma \rangle$ that is not an abelian group with exponent dividing $p-1$, all proper subgroups of $D(p,2)$ are supersoluble but $P \rtimes A$. Note that $P \rtimes A$ is not a $\mathfrak{P}_1$-group. Therefore, every proper $\mathfrak{P}_1$-subgroup of $E(p,q,1)$, $M(p,q,n,\theta,\omega)$, $Q(p,2)$, and $D(p,2)$ is supersoluble.

Proposition 3.5. *Let θ^k and ω^r be a primitive q^nth root and a primitive pth root of unity in F_p, respectively. Then $M(p,q,n,\theta,\omega) \simeq M(p,q,n,\theta^k,\omega^r)$.*

PROOF. Keep the notations of Example 3.2. There exists a natural number a such that $(\theta^k)^a = \theta$ and $(a,q) = 1$ for θ and θ^k are both primitive q^nth roots of unity in $\mathrm{F_p}$. Since

$$M(p,q,n,\theta^k,\omega^r) = \overline{P} \rtimes \langle \overline{\beta}, \overline{\gamma} \rangle = \overline{P} \rtimes \langle (\overline{\beta})^a, \overline{\gamma} \rangle \simeq M(p,q,n,\theta^{ka},\omega^{ra}) = M(p,q,n,\theta,\omega^{ra}),$$

where each of $\overline{P}$, $\overline{\beta}$, and $\overline{\gamma}$ is similar to P, β, and γ, respectively in Example 3.2, it suffices to show that $M(p,q,n,\theta,\omega) \simeq M(p,q,n,\theta,\omega^m)$ for all $1 \leq m \leq q-1$.

Let $P' = \mathrm{F}_p v_1' + \mathrm{F}_p v_2' + \cdots + \mathrm{F}_p v_q'$, where $(v_1', v_2', \ldots, v_q')$ is a basis of P' and

$$\beta' = \begin{pmatrix} \theta & & & \\ & \theta\omega^m & & \\ & & \ddots & \\ & & & \theta\omega^{(q-1)m} \end{pmatrix}_{q\times q}, \quad \gamma' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{q\times q}.$$

Let $Q' = \langle \beta', \gamma' \rangle$ and $G' = P' \rtimes Q' = M(p,q,n,\theta,\omega^m)$. Turning to $M(p,q,n,\theta,\omega)$, let $v_i = v_{r_i}$, where $1 \leq r_i \leq q$ and $q \mid i - r_i$ for all $i > q$. Since $v_i\gamma = v_{i+1}$, we have that $v_i\gamma^m = v_{i+m}$. Note that $(v_1, v_{1+m}, \ldots, v_{1+(q-1)m})$ is a basis of P. Then $P = \mathrm{F}_p v_1 + \mathrm{F}_p v_{1+m} + \cdots + \mathrm{F}_p v_{1+(q-1)m}$ and β, γ^m have the following matrix representation corresponding to the basis $(v_1, v_{1+m}, \ldots, v_{1+(q-1)m})$:

$$\beta = \begin{pmatrix} \theta & & & \\ & \theta\omega^m & & \\ & & \ddots & \\ & & & \theta\omega^{(q-1)m} \end{pmatrix}_{q\times q}, \quad \gamma^m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{q\times q}.$$

Thus

$$M(p,q,n,\theta,\omega) = P \rtimes \langle \beta, \gamma \rangle = P \rtimes \langle \beta, \gamma^m \rangle \simeq P' \rtimes \langle \beta', \gamma' \rangle = M(p,q,n,\theta,\omega^m).$$

Therefore, $M(p,q,n,\theta,\omega) \simeq M(p,q,n,\theta,\omega^m)$ for all $1 \leq m \leq q-1$. This completes the proof. $\square$

By Proposition 3.5, we can briefly denote $M(p,q,n,\theta,\omega)$ by $M(p,q,n)$.

In the rest part of this paper, $\mathfrak{A}(p-1)$ denotes the class of all abelian groups with the exponent dividing $p-1$.

Theorem 3.6. *Assume that G is a $\mathfrak{P}$-group. Then $|\pi(G)| = 2$ and G is isomorphic to one of the groups $E(p,q,1)$, $M(p,q,n)$, $Q(p,2)$, and $D(p,2)$.*

Proof. Since G is a $\mathfrak{P}$-group, G is a nonsupersoluble $\mathfrak{P}_1$-group and every proper $\mathfrak{P}_1$-subgroup and every nontrivial factor group of G is supersoluble. Assume that $G = MN$, where M and N are two normal supersoluble subgroups of G. We proceed the proof in the following steps:

(1) $\Phi(G) = 1$.

Suppose that $\Phi(G) \neq 1$. The assumption on G implies that $G/\Phi(G)$ is supersoluble. So G is supersoluble; a contradiction.

(2) G has the unique minimal normal subgroup K and G/K is supersoluble.

Let K be a minimal normal subgroup of G. The assumption on G implies that G/K is supersoluble. Suppose that H is another minimal normal subgroup of G. Then G/H is supersoluble. This induces that G is supersoluble; a contradiction. Thus uniqueness follows.

(3) Let p be the largest prime in $\pi(G)$. Then G is p-closed and $K = F(G) = O_p(G)$.

M and N are p-closed since M and N are supersoluble. It follows that G is p-closed since M and N are normal subgroups of G and $G = MN$. Therefore, it suffices to show that $K = F(G)$. By (1) and Lemma 2.1, $F(G) = A_1 \times \cdots \times A_k$ where A_i is a minimal normal subgroup of G for $1 \leq i \leq k$. By (2), $k = 1$ and $F(G) = K$. Hence (3) holds.

(4) $M/K \in \mathfrak{A}(p-1)$, $N/K \in \mathfrak{A}(p-1)$ and G/K is a nonabelian q-group with $q \mid p-1$.

By (3), $F(M) = K$. It follows that M/K is abelian since M is supersoluble. By (1), $\Phi(M) = 1$. By Lemma 2.1, $K = A_1 \times \cdots \times A_r$ where A_i, $1 \leq i \leq r$, is a minimal normal subgroup of M. Since M is supersoluble, $M/C_M(A_i) \in \mathfrak{A}(p-1)$, $1 \le i \le r$. Hence, $M/(C_M(A_1) \cap \cdots \cap C_M(A_r)) \in \mathfrak{A}(p-1)$. Note that

$$C_M(A_1) \cap \cdots \cap C_M(A_r) = C_M(K) = C_M(F(M)) \le F(M) = K.$$

Hence $M/K \in \mathfrak{A}(p-1)$. With the same argument, $N/K \in \mathfrak{A}(p-1)$. Assume now that $|\pi(G/K)| \ne 1$. Let $q \in \pi(G/K)$ and let Q_1/K and Q_2/K be the Sylow q-subgroups of M/K and N/K respectively. Clearly, Q_1 and Q_2 are normal supersoluble subgroups of G and $(Q_1Q_2)/K$ is the normal Sylow q-subgroup of G/K. Hence Q_1Q_2 is a proper $\mathfrak{P}_1$-subgroup of G. The assumption on G implies that Q_1Q_2 is supersoluble. It follows that $(Q_1Q_2)/K$ is abelian since $K = F(G) = F(Q_1Q_2)$. Therefore, G/K is abelian since every Sylow subgroup of G/K is normal abelian in G/K. The above shows that G/K is abelian, $M/K \in \mathfrak{A}(p-1)$, and $N/K \in \mathfrak{A}(p-1)$. This implies that $G/K \in \mathfrak{A}(p-1)$. By [19, Chapter 1, Theorem 1.4], K is cyclic. Therefore, G is supersoluble; a contradiction. Thus $|\pi(G/K)| = 1$, and so G/K is a q-group for some prime $q \ne p$. It is clear that G/K is nonabelian and $q \mid p-1$. Hence (4) holds.

(5) Let $M_1 \ge M$ and $N_1 \ge N$ be two maximal normal subgroups of G. Then M_1 and N_1 are supersoluble, while M_1/K and N_1/K are abelian and $1 \ne M_1/K \cap N_1/K$ is cyclic.

Clearly $M_1 = MN \cap M_1 = M(N \cap M_1)$ and M, $N \cap M_1$ are two normal supersoluble subgroups of M_1. The assumption of G implies that M_1 is supersoluble. Then, since $K = F(G) = F(M_1)$, M_1/K is abelian. With the same argument, N_1 is supersoluble and N_1/K is abelian.

Let Q be a Sylow q-subgroup of G. By (3) and (4), $G = KQ$. Since K is a minimal normal subgroup of G and Q is a complement of K in G; therefore, Q is a maximal subgroup of G. Moreover, $M_1 = M_1 \cap KQ = K(Q \cap M_1)$ and $N_1 = N_1 \cap KQ = K(Q \cap N_1)$. Because M_1/K and N_1/K are both abelian, $Q \cap M_1$ and $Q \cap N_1$ are different abelian maximal subgroups of Q. Let $Q' = (Q \cap M_1) \cap (Q \cap N_1)$. If $Q' = 1$ then Q is abelian, and so G/K is abelian; a contradiction. This implies that $1 \ne Q' \le Z(Q)$. We claim that Q' acts regularly on K by conjugation. Assume this false. There exist $1 \ne \alpha \in Q'$ and $1 \ne k \in K$ such that $k^\alpha = k$. Then $C_G(\langle \alpha \rangle) \ge \langle Q, k \rangle = G$ since Q is a maximal subgroup of G. This implies that $\langle \alpha \rangle \trianglelefteq G$, which contradicts (2). Hence Q' acts regularly on K. Since $K = F(G)$, Q' acts faithfully on K. Therefore, Q' can be viewed as a regular group of automorphisms of K. By Lemma 2.3, KQ' is a Frobenius group with complement Q'. By Lemma 2.2, Q' is cyclic. This implies that $M_1/K \cap N_1/K = (M_1 \cap N_1 \cap Q)K/K = Q'K/K \ne 1$ is cyclic. Thus (5) is true.

Without loss of generality, we may assume that M and N are maximal normal subgroups of G. Let Q be a Sylow q-subgroup of G. In the rest part of the proof, Q_1 and Q_2 are fixed q-groups with $Q_1 = M \cap Q$ and $Q_2 = N \cap Q$. Then $M = KQ_1$ and $N = KQ_2$. By (5), Q_1 and Q_2 are abelian maximal subgroups of Q and $1 \ne Q_1 \cap Q_2$ is cyclic. Let $Q_1 \cap Q_2 = \langle \alpha \rangle$. Then $\langle \alpha \rangle \le Z(Q)$ and $|Q/\langle \alpha \rangle| \le q^2$.

(6) $\langle \alpha \rangle = Z(Q)$ and $Q/\langle \alpha \rangle \simeq Z_q \times Z_q$.

If one of the above statements is not true, then $Q/Z(Q)$ is cyclic. This implies that Q is abelian. Then G/K is abelian, which contradicts (4). Hence (6) holds.

Obviously, K is an elementary abelian p-group and K can be viewed as a vector space over F_p. Since $K = F(G)$, $C_G(K) = K$, and Q can be considered as a linear transformation group on K. The conjugate action of Q on K makes K an irreducible Q-module. Since M is supersoluble, K has a basis $\mathscr{M}$ such that any element of Q_1 has a diagonal matrix representation corresponding to $\mathscr{M}$ by the Maschke Theorem [16, Chapter 3, Theorem 3.1]. Then

$$\alpha = \begin{pmatrix} a_1 I_{n_1} & & & \\ & a_2 I_{n_2} & & \\ & & \ddots & \\ & & & a_k I_{n_k} \end{pmatrix}$$

where a_i is distinct from each other for $1 \le i \le k$.

(7) $k = 1$; i.e., α is a scalar linear transformation on K.

Let $V = \{v \mid v \in K, v^{\alpha} = a_1 v\}$. Then V is a nontrivial Q-submodule of K for $\alpha \in Z(Q)$. Since K is an irreducible Q-module, we have that $V = K$. Therefore, (7) is true.

By (6), for every $\beta \in Q_1 \backslash \langle \alpha \rangle$, we have that $\beta^q \in \langle \alpha \rangle$. By (7),

$$\beta = \begin{pmatrix} bI_{m_1} & & & \\ & b\omega^{k_2-1} I_{m_2} & & \\ & & \ddots & \\ & & & b\omega^{k_r-1} I_{m_r} \end{pmatrix}$$

where ω is a primitive qth root of unity in F_p and $1 \leq k_i \neq k_j \leq q-1$ if $i \neq j$.

(8) $r > 1$.

If $r = 1$, then $\beta \in Z(Q) = \langle \alpha \rangle$; a contradiction. Hence (8) holds.

By (8) and $Q_1 = \langle \alpha, \beta \rangle$, K may be considered as a Q_1-module which has r Wedderburn components (see [16, p. 72]) $V_1, V_2, \ldots, V_r$, $r > 1$ (V_i is actually the eigenspace with eigenvalue $b\omega^{k_{i-1}}$ of β).

(9) $r = q$ and $m_1 = m_2 = \cdots = m_q = n$ for some $n \geq 1$.

Let $\gamma \in Q \backslash Q_1$. By the Clifford Theorem [16, Chapter 3, Theorem 4.1], $\langle \gamma \rangle$ induces a transitively permutative action on $\{V_1, V_2, \ldots, V_r\}$ since $Q_1 \trianglelefteq Q$, $Q = Q_1 \langle \gamma \rangle$ and K is an irreducible Q-module. This implies that $m_1 = m_2 = \cdots = m_r = n$ for some $n \geq 1$. Since $\gamma^q \in \langle \alpha \rangle$ by (6) and V_i is Q_1-module for $1 \leq i \leq r$, we have that $r = q$. Thus (9) is true.

Let $\gamma \in Q \backslash Q_1$. From the proof of (9), K has a basis $(v_1^1, \ldots, v_1^n, v_2^1, \ldots, v_2^n, \ldots, v_q^1, \ldots, v_q^n)$ such that α, β, and γ have the following matrix representation corresponding to that basis:

$$\alpha = \begin{pmatrix} aI_n & & & \\ & aI_n & & \\ & & \ddots & \\ & & & aI_n \end{pmatrix}_{qn \times qn}, \quad \beta = \begin{pmatrix} \bar{b}I_n & & & \\ & \bar{b}\omega^{k'_1} I_n & & \\ & & \ddots & \\ & & & \bar{b}\omega^{k'_{q-1}} I_n \end{pmatrix}_{qn \times qn},$$

$$\gamma = \begin{pmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & I_n \\ C & 0 & 0 & \cdots & 0 \end{pmatrix}_{qn \times qn},$$

where $1 \leq k'_i \leq q-1$ and k'_i is distinct from each other for $1 \leq i \leq q-1$.

(10) $n = 1$ and there exists k such that $(k, q) = 1$ and $q \mid k'_i - ik$ for any $1 \leq i \leq q-1$.

Note that $\gamma^q \in \langle \alpha \rangle$. This implies that $C = cI_n$ where $c \in \mathrm{F}_p^*$. Then $\langle v_1^1 \rangle \times \langle v_2^1 \rangle \times \cdots \times \langle v_q^1 \rangle$ is a Q-submodule of K. It follows that $\langle v_1^1 \rangle \times \langle v_2^1 \rangle \times \cdots \times \langle v_q^1 \rangle = K$ and $n = 1$. By (6), $[\beta, \gamma] \in \langle \alpha \rangle$ is of order q since $[\beta, \gamma]^q = [\beta^q, \gamma] = 1$. Directly calculating, we see that there exists k such that $(k, q) = 1$ and $q \mid k'_i - ik$ for any $1 \leq i \leq q-1$. Therefore, (10) is true.

Without loss of generality, assume that $k'_i = i$, for $1 \leq i \leq q-1$, otherwise we can choose another primitive qth root $\omega' = \omega^k$ of unity in F_p.

(11) Conclusion.

Case I. Q_1 and Q_2 are both noncyclic.

In this case, we can choose that $\beta \in Q_1$ and $\gamma \in Q_2$ both of order q. Assume that $|\alpha| > q$. Let $A = K\langle \alpha^q, \beta, \gamma \rangle$, $B = K\langle \alpha^q, \beta \rangle$ and $C = K\langle \alpha^q, \gamma \rangle$. Then $F(A) = K$ and B, C are normal supersoluble subgroups of A. Since $K = F(A)$ and $A/K \simeq \langle \alpha^q, \beta, \gamma \rangle$ is nonabelian, A is nonsupersoluble, which contradicts the assumption on G since A is a proper $\mathfrak{P}_1$-subgroup of G. Hence $|\alpha| = q$. Then α, β, and γ

have the matrix representation

$$\alpha = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega \end{pmatrix}_{q\times q}, \quad \beta = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{q-1} \end{pmatrix}_{q\times q}, \quad \gamma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{q\times q}$$

where ω is a primitive qth root of unity in F_p. Then $G \simeq K \rtimes \langle \alpha, \beta, \gamma \rangle$. Moreover, if $q \neq 2$, then $G \simeq E(p, q, 1)$.

Case II. Q_1 is cyclic and Q_2 is noncyclic or Q_2 is cyclic and Q_1 is noncyclic.

Without loss of generality, we may assume that $Q_1 = \langle \beta \rangle$ and $Q_2 = \langle \alpha \rangle \times \langle \gamma \rangle$ with $|\alpha| = q^n$, $n \geq 1$, and $|\gamma| = q$. Then $|\beta| = q^{n+1}$. So β and γ have the matrix representation:

$$\beta = \begin{pmatrix} \theta & & & \\ & \theta\omega & & \\ & & \ddots & \\ & & & \theta\omega^{q-1} \end{pmatrix}_{q\times q}, \quad \gamma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{q\times q}$$

where θ is a primitive q^{n+1}th root and ω is a primitive qth root of unity in F_p. Hence $G \simeq K \rtimes \langle \beta, \gamma \rangle = M(p, q, n+1)$, $n \geq 1$.

Case III. Q_1 and Q_2 are both cyclic.

Assume that there is $a \in Q \backslash Q_1$ of order q. Note that $\langle \alpha, a \rangle \in \mathfrak{A}(p-1)$ and $[Q, Q] \leq \langle \alpha \rangle$ by (6). Then $A = K\langle \alpha, a \rangle$ is a normal supersoluble subgroup of G and $G = MA$. This can be reduced to Case II since we can replace N by A.

Therefore, we can assume that Q has only one subgroup of order q. Then Q is either a cyclic group or a generalized quaternion group. Since Q is nonabelian, Q is a generalized quaternion group and $q = 2$; i.e.,

$$Q \simeq \langle a, b \mid a^{2^n} = 1,\ a^{2^{n-1}} = b^2,\ a^b = a^{-1} \rangle,$$

where $n \geq 2$. It follows that $|Z(Q)|$=2. This implies that $|Q| = 8$ since $Q/Z(Q) \simeq Z_q \times Z_q$ by (6). Hence $Q \simeq Q_8$ and we can choose β and γ such that $Q_1 = \langle \beta \rangle$ and $Q_2 = \langle \gamma \rangle$, while β and γ have the following matrix representation:

$$\beta = \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where θ is a primitive 4th root of unity in F_p. Then $G \simeq K \rtimes \langle \beta, \gamma \rangle = Q(p, 2)$.

Assume that $q = 2$ in Case I. If $4 \mid p-1$, it can be reduced to Case II since we can put $Q_1 = Q_1' = \langle \beta\gamma \rangle$ and $Q_2 = Q_2' = \langle \alpha, \gamma \rangle$ ($M' = KQ_1'$ and $N' = KQ_2'$ are two normal supersoluble subgroups of G and $G = M'N'$). Then $G \simeq M(p, 2, 2)$. Assume now that $4 \nmid p-1$. Then it can not be reduced to Case II since KQ_1' is nonsupersoluble and $G \simeq D(p, 2)$. Therefore, in Case I, if $q = 2$ then G is isomorphic to one of the groups $M(p, 2, 2)$ and $D(p, 2)$. By the above G is isomorphic to one of the groups $E(p, q, 1)$, $M(p, q, n)$, $Q(p, 2)$, and $D(p, 2)$. This completes the proof. $\square$

4. Some Applications of Theorem 3.6

Let A be a group. G has an *A-factor* if G has subgroups H and K such that $K \trianglelefteq H$ and $H/K \simeq A$.

The following corollary is straightforward from Theorem 3.6, which give a necessary and sufficient condition for a $\mathfrak{P}$ɪ-group to be supersoluble.

Corollary 4.1. *Suppose that $G = MN$ where M and N are normal supersoluble subgroups of G. Then G is supersoluble if and only if G has no A-factor, where A is isomorphic to one of the groups $E(p, q, 1)$, $M(p, q, n)$, $Q(p, 2)$, and $D(p, 2)$.*

Note that the commutator subgroups of $E(p, q, 1)$, $M(p, q, n)$, $Q(p, 2)$, and $D(p, 2)$ are nonnilpotent. By Corollary 4.1, the following corollary is obvious.

Corollary 4.2 [2]. *Suppose that $G = MN$ where M and N are normal supersoluble subgroups of G. Then G is supersoluble if $[G, G]$ is nilpotent.*

Corollary 4.3 [3]. *Suppose that M and N are normal supersoluble subgroups of G. Then G is supersoluble if the indexes of $|G : M|$ and $|G : N|$ are coprime.*

Proof. The coprime indexes of $|G : M|$ and $|G : N|$ imply that $G = MN$. If G is nonsupersoluble, then G has some factor H/K such that H/K is isomorphic to one of the groups $E(p, q, 1)$, $M(p, q, n)$, $Q(p, 2)$, and $D(p, 2)$ by Corollary 4.1. The indexes of $|H : H \cap M|$ and $|H : H \cap N|$ are coprime for $|H : H \bigcap M| = |HM : M|$ and $|H : H \cap N| = |HN : N|$. Therefore, $(H, H \cap M, H \cap N)$ satisfies the assumption. Clearly H is nonsupersoluble . Without loss of generality, we may assume that $H = G$. Note that the indexes of $|G/K : MK/K|$ and $|G/K : NK/K|$ are coprime. It follows that either $MK/K = G/K$ or $NK/K = G/K$ since $|\pi(G/K)| = |\pi(H/K)| = 2$ and $G/K = H/K$ has an unique minimal normal subgroup which is a Sylow subgroup of G/K. This implies that G/K is supersoluble; i.e., H/K is supersoluble. This yields a contradiction. □

If H is an elementary abelian group with $|H| = p^n$ for some prime p, then we call H of rank n and put $r(H) = n$.

Theorem 4.4. *Suppose that $G = MN$, where M and N are normal supersoluble subgroups of G. Let q be the least prime in $\pi(M) \cap \pi(N)$. If $r(H/K) < q$ for every chief factor H/K of G below $M \cap N$, then G is supersoluble.*

Proof. If $\pi(M) \cap \pi(N) = \varnothing$, then $M \cap N = 1$ and $G = M \times N$. In this case, G is supersoluble. Therefore, we can assume that $\pi(M) \cap \pi(N) \neq \varnothing$. Assume that the claim is false and let G be a counterexample such that $|G|$ is minimal. Then

(1) $\Phi(G) \cap M = \Phi(G) \cap N = 1$.

If one of the above statements is not true, then either $\Phi(G) \cap M \neq 1$ or $\Phi(G) \cap N \neq 1$. Without loss of generality, assume that $\Phi(G) \cap M \neq 1$. Let $A = \Phi(G) \cap M$. Since $M/A \bigcap NA/A = (M \cap N)A/A$ and $\pi(M/A) \cap \pi(NA/A) \subseteq \pi(M) \cap \pi(N)$, the hypothesis holds for $(G/A, M/A, NA/A)$. The choice of G implies that G/A is supersoluble. It follows that G is supersoluble; a contradiction.

(2) Let $K \le M$ be a minimal normal subgroup of G. Then K is the unique minimal normal subgroup of G lying in M or N, while $K = F(M) = F(N)$ and G/K is supersoluble.

Since $M/K \bigcap NK/K = (M \cap N)K/K$ and $\pi(M/K) \cap \pi(NK/K) \subseteq \pi(M) \cap \pi(N)$, the hypothesis holds for $(G/K, M/K, NK/K)$. The choice of G implies that G/K is supersoluble. This implies that G has the unique minimal normal subgroup K lying in M. By (1) and Lemma 2.1, $F(M) = K$. Now assume that $A \le N$ is a minimal normal subgroup of G. With the same argument, A is the unique minimal normal subgroup of G lying in N, G/A is supersoluble, and $A = F(N)$. Since G is nonsupersoluble, $A = K$. Therefore, (2) is true.

(3) Let p be the largest prime of $\pi(G)$. Then G is p-closed, $K = O_p(G)$, and $F(G) = K \times Z(G)$.

Since M and N are supersoluble, M and N are p-closed. By (2) and since $G = MN$, $K = O_p(G) = O_p(M)O_p(N)$ is the Sylow p-subgroup of G. Thus G is p-closed and $F(G) = K \times O_{p'}(F(G))$. By (2), $O_{p'}(F(G)) \cap M = O_{p'}(F(G)) \cap N = 1$. Obviously, $K \bigcap Z(G) = 1$. Then $F(G) = K \times Z(G)$. Thus (3) holds.

(4) Let $A \le M$ and let $B \le N$ be two normal subgroups of G. Then the hypothesis holds for (AB, A, B).

Let q' be the least prime in $\pi(A) \cap \pi(B)$ (in case $\pi(A) \cap \pi(B) = \varnothing$, we also say that the hypothesis holds for (AB, A, B) since, in this case, AB is supersoluble). Then $q' \ge q$. If the statement is false, then $A \cap B$ includes a chief factor H/K of AB with $r(H/K) \ge q'$. Note that $A \cap B$ is a normal subgroup of G

contained in $M \cap N$. By the Jordan–Hölder Theorem with operator, $M \cap N$ includes a chief factor H'/K' of G such that $r(H'/K') \geq r(H/K) \geq q' \geq q$, which contradicts the hypothesis. Hence (4) is true.

(5) $M/K \in \mathfrak{A}(p-1)$, $N/K \in \mathfrak{A}(p-1)$ and G/K is a nonabelian q-group and $q \mid p-1$.

By (1), $\Phi(M) = \Phi(N) = 1$. By the same argument as (4) of the proof of Theorem 3.6, $M/K \in \mathfrak{A}(p-1)$ and $N/K \in \mathfrak{A}(p-1)$. Assume that $|\pi(G/K)| \geq 2$. Let $r \in \pi(G/K)$ and let R_1/K and R_2/K be the Sylow r-subgroups of M/K and N/K respectively. Then $R_1 \leq M$ and $R_2 \leq N$ are normal supersoluble subgroups of G and $(R_1R_2)/K$ is the normal Sylow r-subgroup of G/K. By (4), R_1R_2 is supersoluble since R_1R_2 is a proper subgroup of G. Then $R_1R_2Z(G)$ is supersoluble too. It follows that $(R_1R_2)Z(G)/(K \times Z(G))$ is abelian since $F(G) = K \times Z(G) = F(R_1R_2Z(G))$ by (3). Note that $(R_1R_2)Z(G)/(K \times Z(G))$ is a normal Sylow r-subgroup of $G/(K \times Z(G))$. Then $G/(K \times Z(G))$ is abelian since every Sylow subgroup of $G/(K \times Z(G))$ is normal abelian in $G/(K \times Z(G))$. The above shows that $M/K \in \mathfrak{A}(p-1)$, $N/K \in \mathfrak{A}(p-1)$ and $G/(K \times Z(G))$ is abelian. This implies that $G/(K \times Z(G)) \in \mathfrak{A}(p-1)$. Clearly, $(K \times Z(G))/Z(G)$ is a minimal normal subgroup of $G/Z(G)$ and

$$(G/Z(G))/((K \times Z(G))/Z(G)) \simeq G/(K \times Z(G)) \in \mathfrak{A}(p-1).$$

By [19, Chapter 1, Theorem 1.4], $(K \times Z(G))/Z(G)$ is cyclic, and so K is cyclic. Therefore, G is supersoluble; a contradiction. Hence $|\pi(G/K)| = 1$ and G/K is a r-group for some prime $r \neq p$. It is clear that G/K is nonabelian and $r \mid p-1$. If $r = q$, then (5) holds. Hence we can assume that $r \neq q$. This implies that $p = q$ since $\pi(G) = \{p, r\}$. Note that $p > r$. This implies that $r \notin \pi(M) \bigcap \pi(N)$. It follows that either $G = N$ or $G = M$ since K is a Sylow p-subgroup of G lying in M and N by (3) and $G = MN$. Therefore, G is supersoluble; a contradiction. Thus (5) holds.

The final contradiction.

By (5), $\pi(G) = \{p, q\}$. Note that K is a minimal normal subgroup of G lying in $M \cap N$. By hypothesis, $r(K) < q$, i.e., $|K| < p^q$. By (5) and Corollary 4.1, G has a factor H/J such that H/J is isomorphic to one of the groups $E(p, q, 1)$, $M(p, q, n)$, $Q(p, 2)$, and $D(p, 2)$ (if H/J is isomorphic to $Q(p, 2)$ or $D(p, 2)$, then $q = 2$). Note that the Sylow p-subgroups of H/K are of order p^q. This implies that $|K| \geq p^q$ since K is a Sylow p-subgroup of G; a contradiction. The final contradiction completes the proof. □

Let G be a group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, where $p_1 > p_2 > \cdots > p_n$ are primes and $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive integers. Recall that G is said to be a Sylow tower group if G has some normal series $1 = G_0 < G_1 < \cdots < G_n = G$ such that $|G_i/G_{i-1}| = p_i^{\alpha_i}$, $i = 1, \dots, n$.

Corollary 4.5. *Suppose that $G = MN$ where M and N are normal supersoluble subgroups of G. Let q be the least prime in $\pi(M) \cap \pi(N)$. If $r(P/\Phi(P)) < q$ for all Sylow subgroups P of $M \cap N$, then G is supersoluble.*

Proof. If $\pi(M) \cap \pi(N) = \varnothing$, then $G = M \times N$ and G is supersoluble. We can, therefore, assume that $\pi(M) \cap \pi(N) \neq \varnothing$. Note that $M \cap N$ is also a normal Sylow tower subgroup of G, i.e., if $|M \cap N| = p_1^{\beta_1} p_2^{\beta_2} \dots p_n^{\beta_n}$ where $p_1 > p_2 > \cdots > p_n$ and $M \cap N$ has some normal series

$$1 = A_0 < A_1 < \cdots < A_n = M \cap N$$

such that $|A_i/A_{i-1}| = p_i^{\beta_i}$, $i = 1, \dots, n$. Since A_i is a characteristic subgroup of $M \cap N$; therefore, A_i is normal in G, $i = 0, 1, \dots, n$.

Assume that G is nonsupersoluble. Then there exists $1 \leq i \leq n$ such that G/A_i is supersoluble and G/A_{i-1} is nonsupersoluble. Let P_i be a Sylow p_i-subgroup of $M \cap N$. Then $A_i/A_{i-1} \simeq P_i$. Let $H/A_{i-1} = \Phi(A_i/A_{i-1})$. Obviously, H is normal in G and $r(A_i/H) = r(P_i/\Phi(P_i)) < q$. Note that if $\pi(M/H) \cap \pi(N/H) = \varnothing$, then G/H is supersoluble. On the other hand, assume that $\pi(M/H) \cap \pi(N/H) \neq \varnothing$ and let q' be the least prime in $\pi(M/H) \cap \pi(N/H)$. Then $q' \geq q$ and $r(A_i/H) < q'$. Since G/A_i is supersoluble and $r(A_i/H) < q'$; therefore, $(G/H, M/H, N/H)$ satisfies the hypothesis of Theorem 4.4 by the Jordan–Hölder Theorem, and so G/H is supersoluble. In any case, G/H is supersoluble. But since $H/A_{i-1} = \Phi(A_i/A_{i-1}) \leq \Phi(G/A_{i-1})$ and $(G/A_{i-1})/(H/A_{i-1}) \simeq G/H$ is supersoluble, G/A_{i-1} is supersoluble; a contradiction. Hence G is supersoluble. □

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